Relative dispersion in direct cascades of generalized two-dimensional turbulence

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Euromech/Ercoftac - Turbulent cascades II

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Spreading of Lagrangian tracers

Strong motivation from geophysical problems

• Understanding of general features of oceanic/atmospheric circulations from trajectories of drifters/balloons

• Applications: spreading of biological populations and chemical concentrations, rescue operations, ...

Turbulent flow models of oceanographic interest

• Dispersion of particles in time and as a function of pair separation distance?

• Relation with the characteristics of the turbulent flow?

Particle dispersion

N particles, with positions $\boldsymbol{x}_i(t)$, in a given flow field $\boldsymbol{u}(\boldsymbol{x},t)$:

$$\frac{d\boldsymbol{x}_i}{dt} = \boldsymbol{u}(\boldsymbol{x}_i(t),t) \qquad \text{with } i = 1, ..., N$$

Relative dispersion:

$$\langle y^2(t)\rangle = \langle |\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)|^2\rangle = \frac{1}{2N(N-1)} \sum_{i\neq j} |\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)|^2$$

with initial pair separation $|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)| = y_0$

Asymptotic behaviors:

small times: $\langle y^2(t)\rangle - y_0^2 \sim t^2$ (ballistic) [Batchelor, 1950]

large times: $\langle y^2(t)\rangle \sim t$ (diffusive)

Theoretical scaling relations

Relative diffusivity:

$$K_{rel}(t) = \frac{1}{2}\frac{d\langle y^2(t)\rangle}{dt} = \frac{1}{2N(N-1)}\sum_{i\neq j}(\boldsymbol{u}_i - \boldsymbol{u}_j)(\boldsymbol{x}_i - \boldsymbol{x}_j) = \langle \delta\boldsymbol{u}(t)\cdot\boldsymbol{y}(t)\rangle$$

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Dimensionally: $\delta u \sim (kE(k))^{1/2}$ with $k \sim 2\pi/y$

where $E(k)$ is the kinetic energy spectrum of the flow

For a power-law spectrum $E(k) \sim k^{-\beta}$:

$\beta < 3$:

$$K_{rel} \sim \langle y^2\rangle^{(\beta+1)/4} \Rightarrow \langle y^2\rangle \sim t^{4/(3-\beta)}$$

Local dispersion, dependent on scales of size y

$\beta > 3$:

$$K_{rel} \sim \langle y^2\rangle \Rightarrow \langle y^2\rangle \sim e^{at}$$

Nonlocal dispersion, controlled by large scales

Generalized 2D turbulence

q: conserved active tracer

$$\frac{\partial q}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} q = 0, \qquad \boldsymbol{u} = (-\partial_y \psi, \partial_x \psi)$$

$$\hat{q} = -k^{\alpha} \hat{\psi} \qquad \text{in Fourier space}$$

Kinetic energy spectrum:

$$\alpha < 2 \;\Rightarrow\; E(k) \sim k^{-\beta} \sim k^{-(4\alpha+1)/3}, \qquad \beta = (4\alpha + 1)/3$$

$$\alpha \geq 2 \;\Rightarrow\; E(k) \sim k^{1-2\alpha}$$

[Pierrehumbert et al., 1996
Watanabe, Iwayama, 2007]

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[Pierrehumbert et al., 1996
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Quasi-geostrophic (QG) model, $\alpha = 2$:

q is relative vorticity, $\hat{q} = -k^2 \hat{\psi}$, $E(k) \sim k^{-3}$

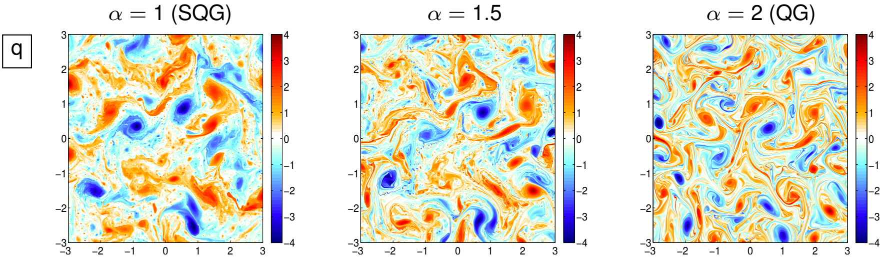
[Kraichnan, 1967]

Surface quasi-geostrophic (SQG) model, $\alpha = 1$:

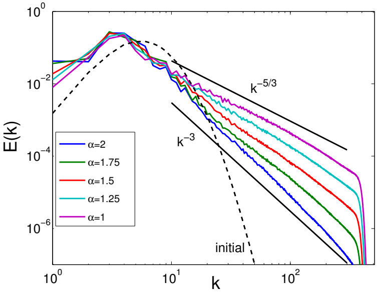
q is surface temperature, $\hat{q} = -k \hat{\psi}$, $E(k) \sim k^{-5/3}$

[Blumen, 1978; Held et al., 1995; Lapeyre, 2017]

Turbulent flow properties

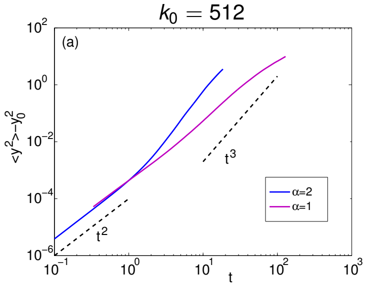


simulations in free decay
[Foussard et al., 2017]



- SQG ($\alpha = 1$):
  eddies from the largest to the smallest scale
  kinetic energy spectrum close to $k^{-5/3}$
- QG ($\alpha = 2$):
  thin quasi-passive filaments
  kinetic energy spectrum steeper than k^{-3}

Time-dependent statistics: relative dispersion



up to $2 \cdot 10^6$ particle pairs
initial separation $y_0 = 2\pi/k_0$

- for small times $\langle y^2(t)\rangle - y_0^2 \sim t^2$
- at intermediate times dispersion grows less rapidly than predicted

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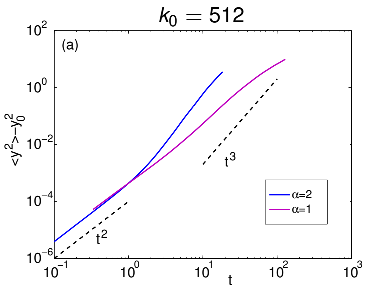

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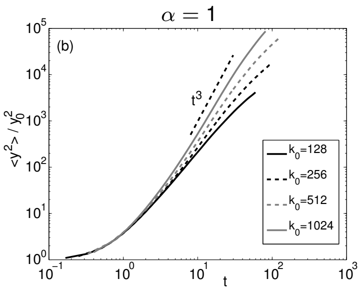
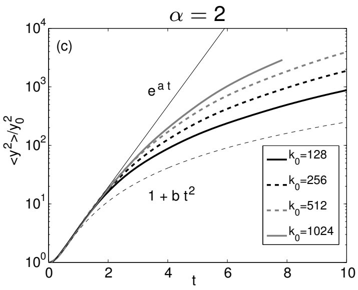
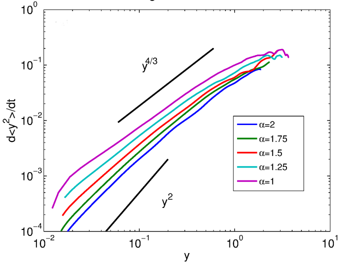
- at intermediate times dispersion grows less rapidly than predicted
- difficulty to observe exponential growth for $\alpha = 2$ (QG)





Relative diffusivity

$k_0 = 512$



$$K_{rel} \propto \frac{d\langle y^2(t)\rangle}{dt} \text{ versus } y = \langle y^2\rangle^{1/2}$$

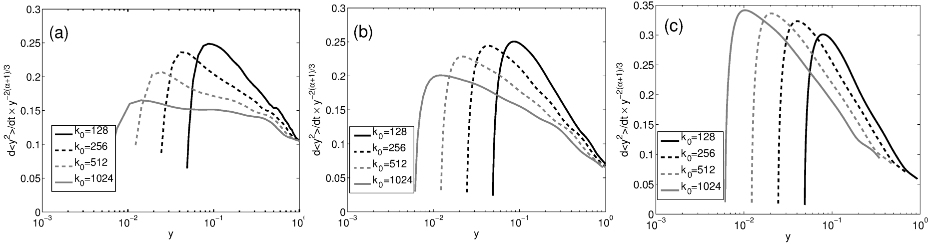
power-law behaviors are more evident

$\alpha = 1$ (SQG): $K_{rel} \sim y^{4/3}$ for small initial separation y_0

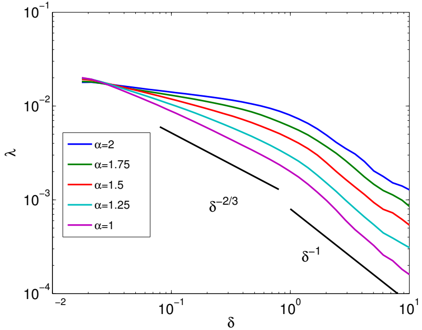
$\alpha = 2$ (QG): $K_{rel} \sim y^2$ difficult to observe even for small y_0

compensated diffusivity $\dfrac{d\langle y^2(t)\rangle}{dt} / \langle y^2\rangle^{(\beta+1)/4}$ with $\beta = (4\alpha + 1)/3$:

$\alpha = 1$ $\alpha = 1.5$ $\alpha = 2$



Finite Size Lyapunov Exponent (FSLE)



$$\lambda(\delta) = \frac{\log r}{\langle \tau(\delta) \rangle}$$

[Aurell et al., 1997; Artale et al., 1997]

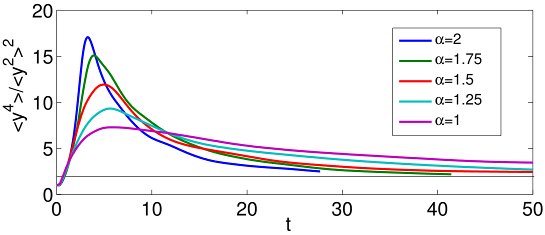
with $\tau(\delta)$ time to observe growth of separation from δ to $r\delta$

$$E(k) \sim k^{-\beta} \Rightarrow \lambda(\delta) \sim \delta^{(\beta-3)/2} \sim \delta^{2(\alpha-2)/3}$$

Expected limiting behaviors: SQG, $\alpha = 1 \Rightarrow \lambda(\delta) \sim \delta^{-2/3}$; QG, $\alpha = 2 \Rightarrow \lambda(\delta) = \text{const}$

- $\alpha < 2$: power-law behavior with exponent in quite good agreement with the theory
- $\alpha = 2$: almost constant FSLE and essentially nonlocal dispersion

Higher order statistics



Kurtosis: $ku(t) = \dfrac{\langle y^4 \rangle}{\langle y^2 \rangle^2}$

$\alpha = 2$: reasonable agreement with exponential growth of separation (for which $ku(t)$ is exponential)

[LaCasce, 2010]

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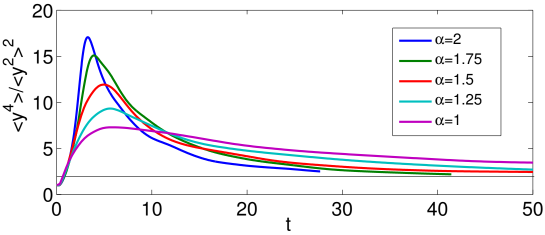

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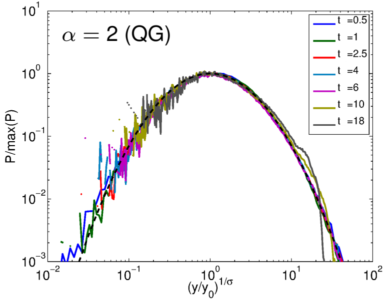


Pair separation PDF:

$\alpha = 2$: consistent with Lundgren PDF

$$P(y,t) \propto \exp\left(-\frac{[\ln(y/y_0) + 2t/T]^2}{4t/T}\right)$$

- Strong support for nonlocal dispersion for the QG model ($\alpha = 2$, $\beta = 3$)

Conclusions

• Relative dispersion statistics in α-turbulence (with QG and SQG models as limiting cases).

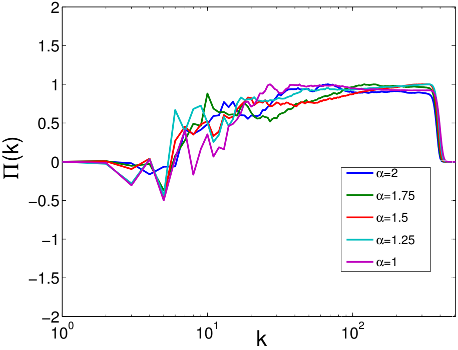
• Strong dependence on initial pair separations.

• For $1 \leq \alpha < 2$ relative dispersion grows in agreement with power-law predictions from local cascade theories. The agreement improves for $\alpha \simeq 1$ and small initial separation.

• For $\alpha = 2$ nonlocal dispersion (exponential growth of separation) is detected by fixed-scale indicators.

A. Foussard, S. Berti, X. Perrot, G. Lapeyre, Journal of Fluid Mechanics **821**, 358 (2017)

Flux of active tracer variance



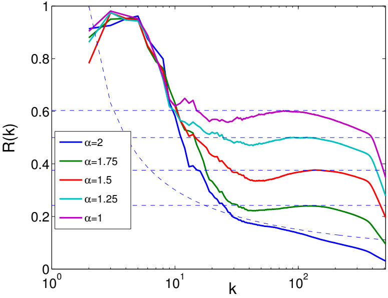
$$\Pi(k) = -\int_k^\infty \left[Re(\widehat{q}^* \, (\widehat{\boldsymbol{u} \cdot \boldsymbol{\nabla} q})) \right] \, p \, dp$$

For all values of α: $\Pi(k) > 0$ for $8 < k < 300$

For most values of α: $\Pi(k) = \text{const}$ between $k^i_{min} = 30$ and $k^i_{max} = 300$

Locality of the strain field

Is scale k strained by eddies of comparable size or by larger ones?



Enstrophy ratio:

$$R(k) = \frac{\int_{k/2}^{k} p^2 E(p) dp}{\int_{k_{min}}^{k} p^2 E(p) dp}$$

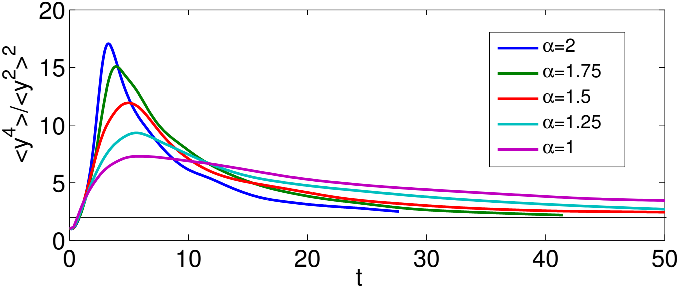
$E(k) \sim k^{-(4\alpha+1)/3} \Rightarrow$ expected behaviors:

$\alpha < 2$: $R(k) = (1 - 2^{4(\alpha-2)/3})/(1 - (k_{min}/k)^{4(2-\alpha)/3})$; $R(k) \to R^{lim} = 1 - 2^{4(\alpha-2)/3}$ for large k

$\alpha = 2$: $R(k) = \log 2/\log(k/k_{min})$

- Larger scales still provide significant stirring for the SQG case

Kurtosis of relative displacements



$$ku(t) = \frac{\langle y^4 \rangle}{\langle y^2 \rangle^2}$$

Rayleigh distribution (diffusive limit of dispersion): $ku = 2$

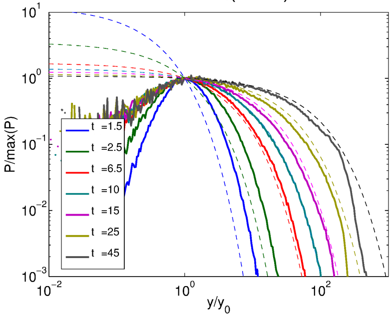
Richardson distribution (local dispersion, for $\alpha = 1$, $\beta = 5/3$): $ku \simeq 5.6$

Lundgren distribution (nonlocal dispersion): ku exponentially grows in time

- For $\alpha = 2$, $ku(t)$ is in reasonable agreement with exponential growth of relative dispersion

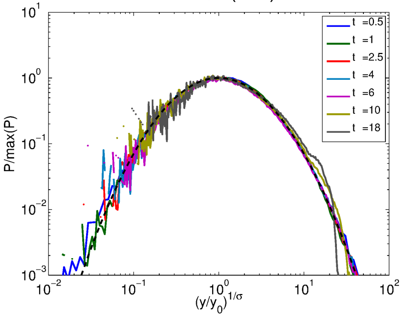
Pair separation PDF

$\alpha = 1$ (SQG)



$$P(y,t) \propto \exp\left(-\frac{9}{4}\frac{y^{2/3}}{\kappa t}\right)$$

$\alpha = 2$ (QG)



$$P(y,t) \propto \exp\left(-\frac{[\ln(y/y_0) + 2t/T]^2}{4t/T}\right)$$

- Separation PDFs are consistent with Lundgren (QG) and Richardson (SQG) PDFs