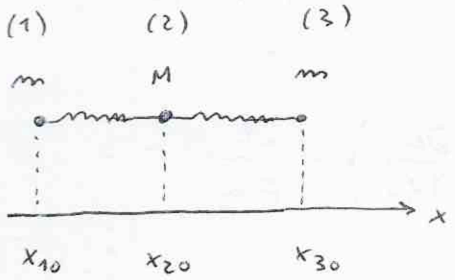
• EX. 1 :



Q1)

$\eta_i = x_i - x_{i0} \qquad i = 1,2,3$ PETITS DEPLACEMENTS

FORCES
$$\begin{cases} F_1 = -k(x_1 - x_{10}) + k(x_2 - x_{20}) = -k\eta_1 + k\eta_2 = k(\eta_2 - \eta_1) \\ F_3 = -k(x_3 - x_{30}) + k(x_2 - x_{20}) = -k\eta_3 + k\eta_2 = k(\eta_2 - \eta_3) \\ F_2 = -F_1 - F_3 = k[(x_1 - x_{10}) - 2(x_2 - x_{20}) + (x_3 - x_{30})] = k(\eta_1 - 2\eta_2 + \eta_3) \end{cases}$$

$$\frac{\partial V}{\partial x_i} = \sum_j \frac{\partial V}{\partial \eta_j} \underbrace{\frac{\partial \eta_j}{\partial x_i}}_{\delta_{ij}} = \frac{\partial V}{\partial \eta_i} \quad ; \quad F_i = -\frac{\partial V}{\partial \eta_i} \quad ; \quad V = V(\eta_1, \eta_2, \eta_3)$$

$$\boxed{V = \frac{k}{2}(\eta_2 - \eta_1)^2 + \frac{k}{2}(\eta_3 - \eta_2)^2}$$
ENERGIE POTENTIELLE

$$\boxed{T = \frac{m}{2}\dot{x}_1^2 + \frac{M}{2}\dot{x}_2^2 + \frac{m}{2}\dot{x}_3^2 = \frac{1}{2}\left(m\dot{\eta}_1^2 + M\dot{\eta}_2^2 + m\dot{\eta}_3^2\right)}$$
ENERGIE CINETIQUE

$\dot{\eta}_i = \dot{x}_i$

LAGRANGIEN
$$\boxed{L = T - V = \frac{1}{2}\left[m\dot{\eta}_1^2 + M\dot{\eta}_2^2 + m\dot{\eta}_3^2 - k(\eta_2 - \eta_1)^2 - k(\eta_3 - \eta_2)^2\right]}$$

Q2)

$$L = \frac{1}{2}\left[m\dot{\eta}_1^2 + M\dot{\eta}_2^2 + m\dot{\eta}_3^2 - k(\eta_2-\eta_1)^2 - k(\eta_3-\eta_2)^2 \right]$$

EQUATIONS de LAGRANGE : $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_i}\right) - \frac{\partial L}{\partial \eta_i} = 0 \qquad i = 1,2,3$

$$\begin{cases} m\ddot{\eta}_1 - k(\eta_2-\eta_1) = 0 \\ M\ddot{\eta}_2 + k(\eta_2-\eta_1) - k(\eta_3-\eta_2) = 0 \\ m\ddot{\eta}_3 + k(\eta_3-\eta_2) = 0 \end{cases}$$

EQUATIONS du MOUVEMENT (PETITES OSCILLATIONS)

$$\sum_{j=1}^{3}\left(T_{ij}\ddot{\eta}_j + V_{ij}\eta_j\right) = 0 \qquad i = 1,2,3$$

$$T = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix} \; ; \; V = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}$$

Q3)

VALEURS PROPRES : $\det(V - \lambda T) = 0 \; ; \quad \lambda \equiv \omega^2$

$$0 = \begin{vmatrix} k - m\lambda & -k & 0 \\ -k & 2k - M\lambda & -k \\ 0 & -k & +k - m\lambda \end{vmatrix} = (k - m\lambda)\left[(2k - M\lambda)(k - m\lambda) - k^2\right] +$$

$$+ k(-k)(k - m\lambda) = (k - m\lambda)\left[(-M\lambda + 2k)(k - m\lambda) - 2k^2\right] =$$

$$= (k - m\lambda)\left[\cancel{2k^2} - 2km\lambda - Mk\lambda + Mm\lambda^2 - \cancel{2k^2}\right] =$$

$$= \lambda(k - m\lambda)(Mm\lambda - 2km - Mk) =$$

$$= \lambda(k - m\lambda)\left[-k(M + 2m) + Mm\lambda\right]$$

$$\Rightarrow \lambda_1 = 0 \; ; \; \lambda_2 = \frac{k}{m} \; ; \; \lambda_3 = \frac{k(M+2m)}{Mm} = \frac{k}{m}\left(1 + \frac{2m}{M}\right) \Rightarrow$$

$$\Rightarrow \boxed{\omega_1 = 0 \; ; \; \omega_2 = \sqrt{\frac{k}{m}} \; ; \; \omega_3 = \sqrt{\frac{k}{m}\left(1 + \frac{2m}{M}\right)}}$$

PULSATIONS PROPRES

Q4)

VECTEURS PROPRES $\underline{a}$: $(V-\lambda T)\underline{a}=0$

$\lambda_1 \equiv \omega_1^2 = 0$

$$\begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}\begin{pmatrix} e_{11} \\ e_{21} \\ e_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} ke_{11} - ke_{21} = 0 & \Rightarrow e_2 = e_{11} \\ -ke_{11} + 2ke_{21} - ke_{31} = 0 & \\ -ke_{21} + ke_{31} = 0 & \Rightarrow e_3 = e_{21} \end{cases}$$

$$\Rightarrow \boxed{\underline{e}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \; ; \; \lambda_1 \equiv \omega_1^2 = 0}$$

$\lambda_2 \equiv \omega_2^2 = \dfrac{k}{m}$

$$\begin{pmatrix} \underbrace{k - \frac{k}{\cancel{m}}\cancel{m}}_{=0} & -k & 0 \\ -k & 2k - \frac{kM}{m} & -k \\ 0 & -k & \underbrace{k - \frac{k}{\cancel{m}}\cancel{m}}_{=0} \end{pmatrix}\begin{pmatrix} e_{12} \\ e_{22} \\ e_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} -ke_{22} = 0 \\ -ke_{12} + \left(2k - \frac{kM}{m}\right)e_{22} - ke_{32} = 0 & \Rightarrow e_{22} = 0 \Rightarrow \\ -ke_{22} = 0 & \quad e_{32} = -e_{12} \end{cases}$$

$$\rightarrow \boxed{\underline{e}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \; ; \; \lambda_2 \equiv \omega_2^2 = \frac{k}{m}}$$

$$\lambda_3 \equiv \omega_3^2 = \frac{k}{m}\left(1+\frac{2m}{M}\right)$$

$$\begin{pmatrix} \underbrace{k - \frac{k}{\cancel{m}}\left(1+\frac{2m}{M}\right)\cancel{m}}^{-\frac{2km}{M}} & -k & 0 \\ -k & \overbrace{2k - \frac{k}{m}\left(1+\frac{2m}{M}\right)M}^{-kM/m} & -k \\ 0 & -k & \underbrace{k - \frac{k}{\cancel{m}}\left(1+\frac{2m}{M}\right)\cancel{m}}_{-\frac{2km}{M}} \end{pmatrix} \begin{pmatrix} e_{13} \\ e_{23} \\ e_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} -\frac{2km}{M} e_{13} - k e_{23} = 0 & \Rightarrow e_{23} = -e_{13}\, 2m/M \\ -k e_{13} - \frac{kM}{m} e_{23} - k e_{33} = 0 & \Rightarrow \\ -k e_{23} - \frac{2km}{M} e_{33} = 0 & \Rightarrow e_{33} = -e_{23}\frac{M}{2m} = + e_{13} \Rightarrow e_{33} = e_{13} \end{cases}$$

$$\Rightarrow \boxed{\underline{e}_3 = \begin{pmatrix} 1 \\ -\frac{2m}{M} \\ 1 \end{pmatrix} ; \ \lambda_3 \equiv \omega_3^2 = \frac{k}{m}\left(1+\frac{2m}{M}\right)}$$

Q5) MODE MOU : $\underline{e}_1 = (1,1,1) \leftrightarrow \omega_1 = 0$

$\omega_1 = 0 \Rightarrow e_{11} = e_{21} = e_{31}$

$\eta = \underline{e}_1 e^{i\omega_1 t}$ → SUBSTITUTION dans EQ. LAGRANGE :

$$\begin{cases} m\ddot{\eta}_1 = k(\eta_2 - \eta_1) = 0 & \leftarrow \eta_2 = \eta_1 \\ M\ddot{\eta}_2 = k(\eta_1 - \eta_2) + k(\eta_3 - \eta_2) = 0 & \leftarrow \eta_2 = \eta_1 = \eta_3 \\ m\ddot{\eta}_3 = k(\eta_2 - \eta_3) = 0 & \leftarrow \eta_3 = \eta_2 \end{cases} \Rightarrow \ddot{\eta}_i = 0 \ \ \forall i \Rightarrow$$

(TRANSLATION UNIFORME)

$\Rightarrow \dot{\eta}_i = \text{const} \ \ \forall i \Rightarrow P = m\dot{\eta}_1 + M\dot{\eta}_2 + m\dot{\eta}_3 = \text{const.}$ CONSERVATION MOMENT LINÉAIRE

CONSÉQUENCE de l'INVARIANCE du LAGRANGIEN sous une TRANSLATION (en ESPACE)

SOLUTION MODE MOU : $\eta_1(t) = \eta_2(t) = \eta_3(t) = at + b$ Tout le système se déplace à la même vitesse constante.

=> Cette solution n'est pas intéressante du point de vue physique, parce que il est toujours possible de travailler dans un référentiel par lequel le moment linéaire total est nul. Dans ce référentiel, le mode mou correspond à la solution d'équilibre.

Q6) $\omega = \omega_2$ => $\underline{e}_2 = (1, 0, -1)$; $\eta = \underline{e}_2 e^{i\omega_2 t}$

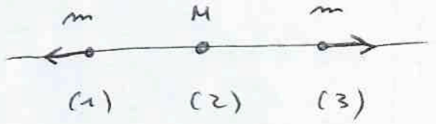
$e_{32} = -e_{12}; e_{22} = 0$

=> si on substitue cette solution dans les EQ. LAGRANGE :

$$\begin{cases} m\ddot{\eta}_1 = k(\cancel{\eta_2} - \eta_1) \overset{\eta_2 = 0}{=} -k\eta_1 \\ M\ddot{\eta}_2 = k(\eta_1 - \cancel{\eta_2}) + k(\eta_3 - \cancel{\eta_2}) \overset{\eta_3 = -\eta_1}{=} 0 \\ m\ddot{\eta}_3 = k(\cancel{\eta_2} - \eta_3) \underset{\eta_2 = 0}{=} -k\eta_3 \underset{\eta_3 = -\eta_1}{=} k\eta_1 \end{cases}$$

M : AU REPOS

les 2 autres masses vibrent avec la même amplitude et des phases opposées.



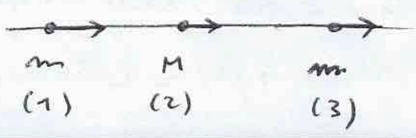
$\omega = \omega_3$ => $\underline{e}_3 = (1, -\frac{2m}{M}, 1)$; $\eta = \underline{e}_3 e^{i\omega_3 t}$

$e_{33} = e_{13}; e_{23} = -\frac{2m}{M}$

=> EQ. LAGRANGE :

$$\begin{cases} m\ddot{\eta}_1 = k(\eta_2 - \eta_1) \overset{\eta_2 = -\frac{2m}{M}\eta_1}{=} -\left(\frac{2m}{M} + 1\right)\eta_1 \\ M\ddot{\eta}_2 = k(\eta_1 - \eta_2) + k(\eta_3 - \eta_2) \overset{\eta_3 = -\eta_1}{=} -k(\eta_2 - \cancel{\eta_1} + \eta_2 + \cancel{\eta_1}) \overset{\eta_2 = -\frac{2m}{M}\eta_1}{=} \frac{2m}{M}k\eta_1 \\ m\ddot{\eta}_3 = k(\eta_2 - \eta_3) \underset{\eta_3 = \eta_1}{=} k(\eta_2 - \eta_1) \underset{\eta_2 = -\frac{2m}{M}\eta_1}{=} -\left(\frac{2m}{M} + 1\right)\eta_1 \end{cases}$$

Les 2 masses m vibrent en phase ; la masse M vibre avec une phase opposée et une amplitude différente



Q7) SOLUTION GÉNÉRALE :

COMBINAISON des MODES PROPRES

$$\begin{pmatrix} \eta_1(t) \\ \eta_2(t) \\ \eta_3(t) \end{pmatrix} = (a + bt)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + B\, e^{i\sqrt{\frac{k}{m}}\,t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + C\, e^{i\sqrt{\frac{k}{m}\left(1+\frac{2m}{M}\right)}\,t}\begin{pmatrix} 1 \\ -\frac{2m}{M} \\ 1 \end{pmatrix}$$

$\downarrow$

SOLUTION PHYSIQUE : $Re[\eta]$

$\eta = (\eta_1, \eta_2, \eta_3)$

Cette solution décrit une vibration longitudinale générique de la molécule. Toute vibration ne faisant pas intervenir une translation rigide sera une combinaison linéaire des modes ω_2 et ω_3.

Les amplitudes des modes normaux, ainsi que leurs phases seront déterminées par les conditions initiales.