EX. : OSCILLATEUR HARMONIQUE 1D

$l_0 = 0$

PFD : $m\underline{a} = \underline{f}$

$\underline{a} = (\ddot{x}, 0, 0)$

$\underline{f} = -k\underline{x}$

$\underline{f} = (f_x, 0, 0) = (-kx, 0, 0)$

Q1)

$$\underline{f} = -\underline{\nabla} V = -\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right)$$

$$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial z} = 0 \quad \text{car } f_y = f_z = 0$$

$$f_x = -kx \Rightarrow V = \frac{kx^2}{2}$$
$$= -\frac{\partial V}{\partial x}$$

$$T = \frac{mv^2}{2} = \frac{m\dot{x}^2}{2}$$

$$L = T - V = \frac{m\dot{x}^2}{2} - \frac{kx^2}{2} \qquad \text{LAGRANGIEN}$$

Q2)

$$H = \left[p\dot{x} - L(x, \dot{x})\right]_{\dot{x} = \dot{x}(p)}$$

TR. LEGENDRE

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \text{MOMENT CANONIQUE CONJUGUÉ}$$

$$\Rightarrow H = p\frac{p}{m} - \frac{\cancel{m}}{2}\frac{p^2}{m^{\cancel{2}}} + \frac{kx^2}{2} = \frac{p^2}{2m} + \frac{kx^2}{2}$$

$$\dot{x} = \frac{p}{m}$$

$$\omega^2 = \frac{k}{m} \Rightarrow H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \Leftrightarrow \boxed{H = \frac{1}{2m}\left(p^2 + m^2\omega^2 x^2\right)}$$

HAMILTONIEN

Q3) ANALYSE DIMENSIONNELLE

$$\left[\frac{p^2}{2m}\right] = \left[\frac{M^{\cancel{2}}\dot{x}^2}{\cancel{M}}\right] = \left[M\dot{x}^2\right] = \left[M\underbrace{L^2t^{-2}}_{v^2}\right] = \left[\underbrace{MLt^{-2}}_{F}\cdot L\right] \qquad \text{ÉNERGIE (CINÉTIQUE)}$$

$$[p] = [M\dot{x}]$$

$$\left[\frac{m^2\omega^2x^2}{2m}\right] = \left[\frac{M^{\cancel{2}}\omega^2L^2}{\cancel{M}}\right] = \left[ML^2t^{-2}\right] = [F\cdot L] \qquad \text{ÉNERGIE (POTENTIELLE)}$$

car associée à la force élastique

$$[\omega^2] = \left[\frac{k}{m}\right] = \left[\frac{F}{LM}\right] = \left[\frac{\cancel{ML}t^{-2}}{\cancel{ML}}\right] = [t^{-2}] \Rightarrow [\omega] = [t^{-1}] \qquad \text{PULSATION}$$

Q4) EQ. HAMILTON

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p} = \dfrac{p}{m} \\ \dot{p} = -\dfrac{\partial H}{\partial x} = -m\omega^2 x \end{cases}$$

Q5)

PFD (vérification) : $m\ddot{x} = m\dfrac{d}{dt}\dot{x} = m\dfrac{d}{dt}\left(\dfrac{p}{m}\right) = \dot{p} = -m\omega^2 x = -kx$

$\Leftrightarrow m\ddot{x} = -kx$

Q6) $H = T + V$ ÉNERGIE MÉCANIQUE

$\dfrac{dH}{dt} = \dfrac{\partial H}{\partial t} \Rightarrow H = E = \text{const}$ CONSERVATION ÉNERGIE MÉCANIQUE

système hamiltonien

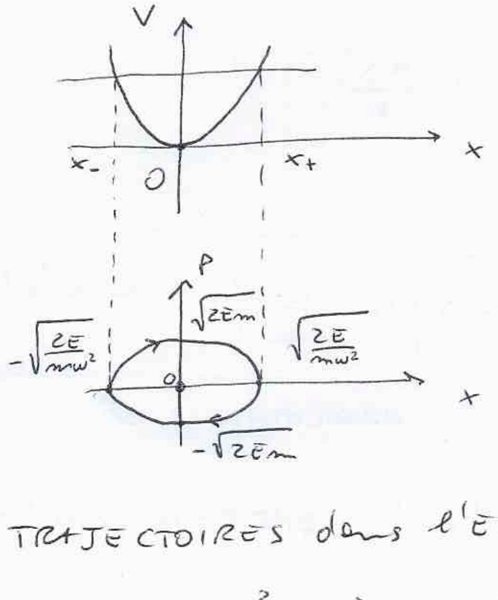
Q7)

$H = T + V = \dfrac{p^2}{2m} + \dfrac{m\omega^2 x^2}{2}$; $V = \dfrac{kx^2}{2} = \dfrac{m\omega^2 x^2}{2}$

EQUILIBRE : $\dfrac{\partial V}{\partial x} = 0 \Rightarrow kx = 0 \Rightarrow x_s = 0$

STABLE car $x_s = 0$ est un MINIMUM de $V(x)$

Q8) PORTRAIT de PHASE



(i) $E = 0 = \min(V) \Rightarrow$ EQUILIBRE

$x = x_s = 0$ avec $p = m\dot{x} = 0 \Rightarrow \dot{x} = 0$

(ii) $E > 0 \Rightarrow$ MOUVEMENT LIMITÉ ($x_- \leq x \leq x_+$)

OSCILLATIONS AUTOUR de la POSITION d'ÉQUILIBRE x_s.

Q9) TRAJECTOIRES dans l'ESPACE des PHASES ; $E > 0$

$x = 0 \Rightarrow H = E = \dfrac{p^2}{2m} \Rightarrow p = \pm\sqrt{2Em}$

$p = 0 \Rightarrow H = E = \dfrac{m\omega^2 x^2}{2} \Rightarrow x = \pm\sqrt{\dfrac{2E}{m\omega^2}}$

$H = E = \dfrac{p^2}{2m} + \dfrac{m\omega^2 x^2}{2} \Rightarrow \boxed{\dfrac{p^2}{2mE} + \dfrac{x^2}{2E/(m\omega^2)} = 1} \Leftrightarrow$ ELLIPSE : $\left(\dfrac{x}{a}\right)^2 + \left(\dfrac{p}{b}\right)^2 = 1$

avec : $b = \sqrt{2Em}$

$a = \sqrt{\dfrac{2E}{m\omega^2}}$

Q10)

AIRE de l'ELLIPSE : $S = \pi a b = \pi \sqrt{\cancel{2m} E \frac{2E}{\cancel{m}\omega^2}} = \frac{2\pi}{\omega} E = T \cdot E$

$T = \frac{2\pi}{\omega}$: PÉRIODE

$= 2\pi \sqrt{\frac{m}{k}}$

D'autre part : $S = \oint p\,dx$ ACTION

$\Rightarrow S = T \cdot E = \text{const.}$ CONSERVATION de l'ACTION

ACTION PÉRIODE ÉNERGIE

Q11) $(x, p) \to (\vartheta, I)$ $\quad I \equiv \frac{S}{2\pi} = \frac{E}{\omega}$

$$\begin{cases} x = \sqrt{\frac{2I}{m\omega}} \sin\vartheta \\ p = \sqrt{2Im\omega} \cos\vartheta \end{cases}$$

si $[x, p]_{(\vartheta, I)} \equiv \frac{\partial x}{\partial \vartheta}\frac{\partial p}{\partial I} - \frac{\partial x}{\partial I}\frac{\partial p}{\partial \vartheta} = 1 \Rightarrow$ TR. CANONIQUE

$$[x,p]_{(\vartheta, I)} = \sqrt{\frac{2I}{m\omega}} \frac{\sqrt{2m\omega}}{2\sqrt{I}} \cos^2\vartheta + \sqrt{\frac{2}{m\omega}} \frac{1}{2\sqrt{I}} \sqrt{2Im\omega} \sin^2\vartheta =$$

$= \cos^2\vartheta + \sin^2\vartheta = 1 \Rightarrow$ la TRANSFORMATION est CANONIQUE

Q12) $K = \frac{1}{\cancel{2m}} \cancel{2m} I \omega \cos^2\vartheta + \frac{1}{2} \cancel{m}\omega^2 \frac{2I}{\cancel{m\omega}} \sin^2\vartheta =$

$= I\omega(\cos^2\vartheta + \sin^2\vartheta) = I\omega \Rightarrow \boxed{K = I\omega}$ HAMILTONIEN

Q13) ϑ : CYCLIQUE $\Rightarrow I = \text{const.}$

Q14) EQ. HAMILTON

$$\begin{cases} \dot{\vartheta} = \frac{\partial K}{\partial I} = \omega \\ \dot{I} = -\frac{\partial K}{\partial \vartheta} = 0 \end{cases}$$

Q15) $I = \text{const}$

$\vartheta = \vartheta_0 + \omega t$ avec $\vartheta_0 = \vartheta(0)$

Q16) Dans les variables originales (x, p) :

en utilisant aussi : $I = \frac{E}{\omega}$

$$\begin{cases} x = \sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + \vartheta_0) \\ p = \sqrt{2Em} \cos(\omega t + \vartheta_0) \end{cases}$$

$$\frac{x^2}{\frac{2E}{m\omega^2}} + \frac{p^2}{2Em} = 1$$

ELLIPSE avec : $a = \sqrt{\frac{2E}{m\omega^2}}$

$b = \sqrt{2Em}$